ACTION OF A CONCENTRATED FORCE ON A CYLINDRICAL SHELL REINFORCED BY A STRINGER

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Studies discussed in the survey [1] resolved questions relating to the transmission of local loads to shells through stiffening ribs. In most of these studies, the complexity of the solutions made it necessary to obtain numerical results on a computer and present them in the form of graphs and tables. Thus, from the viewpoint of constructing simple analytical solutions, the problem requires further study.

Below, we use semi-momentless shell theory to construct compact formulas to calculate the contact reactions, deflections, and bending moments in the rib. An interesting aspect of our approach is the rapid decay of local perturbations in the circumferential direction [2], i.e., the shell is assumed to be infinite in the direction. The solutions are not periodic with respect to the angular coordinate. The use of nonperiodic solutions in problems concerning the local strength of shells takes it impetus from the works of W. Finsterwald, F. Odquist, and other foreign researchers. Meanwhile, an analysis of their errors allowed Chernyshev [3] to choose which simplifying assumptions were suitable for use.

1. Expansion of Components of the Stress—Strain State into Single Trigonometric Series. Let a shell of radius R and thickness h be simply supported by its ends $x = 0$, $x = \ell$ (see Fig. 1). The section $x = x_1$ of the shell is loaded by the normal force P through a rib with the bending stiffness E_1I_1. We need to determine the effect of the shell on the deformation and stress state of the rib, which takes up the applied load.

Besides the above-noted simplifications, we will solve the problem by additionally assuming that the thin-walled cylinder has a small relative elongation ($R \le \ell \le 5R$) and interacts with the rib only by means of normal contact reactions q(x) distributed over the zero generatrix $\varphi = 0$. We will ignore the tangential component of the interaction and the eccentricity of the reinforcement. We will also ignore the mutual effect of several ribs affixed to the cylindrical surface.

Given these assumptions, it is expedient to use the well known solution in [4] as the Green's function. This solution, constructed on the basis of the equations of semi-momentless theory [5], has the form:

$$w(x, x_1) = A \sum_{k=1}^{\infty} k^{-3/2} \sin(kX_1) \sin(kX),$$

$$A = \frac{\sqrt{2-\sqrt{2}}\,[12(1-\nu^2)]^{5/8} R^{3/4} l^{1/2}}{(2\pi)^{3/2} E h^{9/4}}, \quad X_1 = \frac{\pi x_1}{l}, \quad X = \frac{\pi x}{l}. \tag{1.1}$$

Here, $w(x, x_1)$ is the deflection of the zero generatrix at the point (x, 0) from the action of a single normal force at the point $(x_1, 0)$; E and ν are the elastic modulus and the Poisson's ratio of the shell material.

The contact load

$$q(x) = 2l^{-1} \sum_{k=1}^{\infty} a_k \sin(kX) \tag{1.2}$$

causes the displacements

$$w(x) = A \sum_{k=1}^{\infty} a_k k^{-3/2} \sin(kX). \tag{1.3}$$

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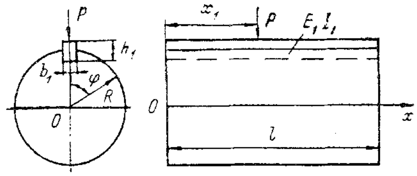

Fig. 1

We take the following as the Green's function for the rib

$$w_1(x, x_1) = \frac{2l^3}{\pi^4 E_1 I_1}\sum_{k=1}^{\infty} k^{-4}\sin(kX_1)\sin(kX), \quad I_1 = \frac{b_1 h_1^3}{12}, \tag{1.4}$$

where $w_1(x, x_1)$ is the deflection of the rib in the section x due to the unit force in the section x_1; b_1 and h_1 are its width and height, measured along a normal to the shell; E_1 is the elastic modulus of the reinforcement.

In accordance with (1.4), the rib deflections from the contact load q(x) and the external force P will be

$$w_1(x) = \frac{2l^3}{\pi^4 E_1 I_1}\sum_{k=1}^{\infty} k^{-4}(P\sin(kX_1) - a_k)\sin(kX). \tag{1.5}$$

The strain compatibility condition $w(x) = w_1(x)$ is satisfied on the contact line (φ = 0). Thus, with allowance for Eqs. (1.3) and (1.5), we obtain

$$A\,a_k k^{-3/2} = \frac{2l^3 k^{-4}}{\pi^4 E_1 I_1}(P\sin(kX_1) - a_k). \tag{1.6}$$

The solution of Eq. (1.6) is

$$a_k = Pc(k^{5/2} + c)^{-1}\sin(kX_1), \; c = 2l^3(\pi^4 E_1 I_1 A)^{-1}. \tag{1.7}$$

Inserting these values of a_k into (1.5), we obtain the following for the deflections of the rib (and the shell along its generatrix)

$$w(x, x_1) = 2Pl^3(\pi^4 E_1 I_1)^{-1} S_1\left(\frac{x}{l}, \frac{x_1}{l}\right),$$
$$S_1\left(\frac{x}{l}, \frac{x_1}{l}\right) = \sum_{k=1}^{\infty}(k^4 + ck^{3/2})^{-1}\sin(kX_1)\sin(kX). \tag{1.8}$$

The bending moment in the rib is proportional to the second derivative of the deflection:

$$M(x, x_1) = 2Pl\pi^{-2}S_2\left(\frac{x}{l}, \frac{x_1}{l}\right),$$
$$S_2\left(\frac{x}{l}, \frac{x_1}{l}\right) = \sum_{k=1}^{\infty} k^{1/2}(k^{5/2} + c)^{-1}\sin(kX_1)\sin(kX). \tag{1.9}$$

We then use (1.2) and (1.7) to obtain a series to calculate the contact reactions:

$$q(x, x_1) = 2Pcl^{-1}S_3\left(\frac{x}{l}, \frac{x_1}{l}\right), \tag{1.10}$$
$$S_3\left(\frac{x}{l}, \frac{x_1}{l}\right) = \sum_{k=1}^{\infty}(k^{5/2} + c)^{-1}\sin(kX_1)\sin(kX).$$

The parameter c determines the effect of the shell on the bending of the rib. In the limit at $c \to 0$, the contact forces $q(x, x_1) = 0$, while Eqs. (1.8), (1.9) describe the bending of a supported team. The other ideal case ($c \to \infty$) corresponds to the loading of an unreinforced shell. Equation (1.8) becomes solution (1.1), while it follows from (1.9)-(1.10) that

$$M(x, x_1) = 0,\ q(x, x_1) = P\delta(x - x_1)$$

[$\delta(z)$ is the Dirac delta function]. Thus, with large c, the contact reactions will be highly localized in the neighborhood of the point of application of the external force.

In the design of refined shells, the value of the parameter c may vary broadly ($0 < c < 300$). The convergence of the series (1.8)-(1.10) slows with an increase in c, so it is best to find a way to speed it up.

2. Speeding up the Convergence of the Solutions. We will use the method developed by A. N. Krylov. Here, before the calculations are performed, it is necessary to isolate and analytically sum the slowly converging part of the series. The following series converge slowly in solution (1.8)-(1.10)

$$\sum_{k=1}^{\infty} k^{-\alpha} \sin(kX_1)\sin(kX) \quad \left(\alpha = 2, \frac{5}{2}, 4\right).$$

We will examine the auxiliary series

$$\sum_{k=1}^{\infty} k^{-\alpha} \cos \frac{k\pi t}{l} \quad (0 \leqslant t \leqslant l).$$

We will subject this series to the Mellin transform. Using tables of integral transforms in [6] (p. 279), we find

$$\int_0^{\infty} t^{s-1} \sum_{k=1}^{\infty} k^{-\alpha} \cos\frac{k\pi t}{l}\, dt = \Gamma(s) \cos\frac{\pi s}{2}\left(\frac{l}{2\pi}\right)^s \sum_{k=1}^{\infty} k^{-\alpha-s} \quad (0 < \operatorname{Re} s < 1) \tag{2.1}$$

where $\Gamma(z)$ is the gamma function.

The series in the right side of (2.1) reduces to the Riemann zeta function [7]

$$\sum_{k=1}^{\infty} k^{-z} = \zeta(z) \quad (\operatorname{Re} z > 1).$$

Thus, having used the inverse Mellin transform, we obtain

$$\sum_{k=1}^{\infty} k^{-\alpha} \cos(k\beta) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \Gamma(s) \cos\left(\frac{\pi s}{2}\right) \beta^{-s} \zeta(\alpha + s)\, ds,$$

$$\beta = \pi t l^{-1},\ i = \sqrt{-1},\ 0 < \sigma < 1.$$

This integral can be calculated from the residue theorem. We will limit ourselves to the case $\alpha \neq 2m + 1$ ($m = 0, 1, 2, \ldots$), when the integrand has only simple poles. For the gamma function, they are located at the points $s = -m$, where $\operatorname*{res}_{s=-m} \Gamma(s) = (-1)^m/m!$ [8]. The Reimann zeta function has a simple pole at $\alpha + s = 1$, and its residue is equal to unity [8]. We reduce integration along a straight line parallel to the imaginary axis to integration over the closed contour formed by an arc of a circle of radius r and the straight line $\operatorname{Re} s = \sigma$. Considering that the integral over the circle arc is equal to zero at $r \to \infty$, we have

$$\sum_{k=1}^{\infty} k^{-\alpha} \cos(k\beta) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \cos\frac{m\pi}{2} \zeta(\alpha - m) \beta^m + \tag{2.2}$$

$$+\Gamma(1-\alpha)\cos\left(\frac{\pi}{2}(1-\alpha)\right)\beta^{\alpha-1}=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\zeta(\alpha-2n)\beta^{2n}+\pi\beta^{\alpha-1}\left(2\Gamma(\alpha)\cos\frac{\pi\alpha}{2}\right)^{-1}.$$

The Riemann zeta function was tabulated in [7]. Meanwhile, $\zeta(-2m) = 0$ for natural numbers. Thus, in the case of even α, the series in the right side of Eq. (2.2) breaks off and reduces to a finite sum. Then assuming that $\alpha = 2$ and $\alpha = 4$, we can use Eq. (2.2) to find

$$\sum_{k=1}^{\infty}k^{-2}\cos(k\beta)=\frac{\pi^2}{2}\left(\frac{1}{3}-\gamma+\frac{1}{2}\gamma^2\right),\quad \gamma=\frac{\beta}{\pi},$$
$$\sum_{k=1}^{\infty}k^{-4}\cos(k\beta)=\pi^4\left(\frac{1}{90}-\frac{1}{12}\gamma^2+\frac{1}{12}\gamma^3-\frac{1}{48}\gamma^4\right), \tag{2.3}$$

which agrees with the well known results in [9]. Series (2.2) does not break off at $\alpha = 5/2$ but does converge quite slowly, since $0 \le \gamma \le 1$. Retaining terms of the order γ^6 in the expansion, we obtain

$$\sum_{k=1}^{\infty}k^{-5/2}\cos(k\beta)=1{,}3414-9{,}3052\gamma^{3/2}+7{,}2047\gamma^2-0{,}1021\gamma^4-0{,}0059\gamma^6+O(\gamma^8). \tag{2.4}$$

Taking Eqs. (2.3), (2.4) into account, we accelerate the convergence of series (1.8)-(1.10) and write them in the form

$$S_1\left(\frac{x}{l},\frac{x_1}{l}\right)=\frac{\pi^4}{24}\left[X_+^2-X_-^2+\frac{1}{4}(X_+^4-X_-^4)+X_-^3-X_+^3\right]-$$
$$-\sum_{k=1}^{\infty}k^{-4}(1+k^{5/2}c^{-1})^{-1}\sin(kX_1)\sin(kX), \tag{2.5}$$
$$S_2\left(\frac{x}{l},\frac{x_1}{l}\right)=\frac{\pi^2}{4}\left[X_+-X_-+\frac{1}{2}(X_-^2-X_+^2)\right]-$$
$$-\sum_{k=1}^{\infty}k^{-2}(1+k^{5/2}c^{-1})^{-1}\sin(kX_1)\sin(kX),$$
$$S_3\left(\frac{x}{l},\frac{x_1}{l}\right)=4{,}6526(X_+^{3/2}-X_-^{3/2})-3{,}6024(X_+^2-X_-^2)+0{,}0511(X_+^4-X_-^4)+$$
$$+0{,}0030(X_+^6-X_-^6)-\sum_{k=1}^{\infty}k^{-5/2}(1+k^{5/2}c^{-1})^{-1}\sin(kX_1)\sin(kX),\quad X_\pm=\frac{|x_1\pm x|}{l}.$$

Table 1 shows values of the functions $S_j(x/\ell, 1/2)$, $j = \overline{1, 3}$, calculated for different x/ℓ and c. They characterize the distribution of the deflections, moments, and contact forces along the rib with the action of an external force in the midsection. An analysis shows that these quantities decrease more rapidly with an increase in the parameter c. The deflections decrease the most slowly. As regards the bending moments and contact forces, they not only decrease with increasing distance from the concentrated external force, but they may change sign near the edge of the rib.

The sought quantities reach their maximum values when the concentrated force is applied in the middle of the shell. It is interesting to attempt to obtain closed formulas to calculate these values.

3. Closed Formulas for the Solutions. We will distinguish two cases: small and large c. At small c, series (2.5) converge fairly rapidly, and keeping several of the initial terms ensures good accuracy. In this case,

$$S_1\left(\frac{1}{2},\frac{1}{2}\right)=\frac{\pi^4}{96}-\frac{1}{1+d}-\frac{1}{81(1+9\sqrt{3}\,d)}-\frac{1}{625(1+25\sqrt{5}\,d)},$$
$$S_2\left(\frac{1}{2},\frac{1}{2}\right)=\frac{\pi^2}{8}-\frac{1}{1+d}-\frac{1}{9(1+9\sqrt{3}\,d)}-\frac{1}{25(1+25\sqrt{5}\,d)}- \tag{3.1}$$

TABLE 1

c	$10x l^{-1}$	$10^4 S_1\left(\frac{x}{l}, \frac{1}{2}\right)$	$10^4 S_2\left(\frac{x}{l}, \frac{1}{2}\right)$	$10^4 S_3\left(\frac{x}{l}, \frac{1}{2}\right)$
10	5	1005	2727	1666
	4	905	984	997
	3	698	242	482
	2	465	−6	207
	1	231	−4	73
30	5	382	1724	859
	4	328	277	348
	3	238	−63	96
	2	152	−85	19
	1	74	−45	2
50	5	240	1399	632
	4	201	110	196
	3	141	−82	36
	2	89	−62	3
	1	43	−27	−1

TABLE 2

c	$10^4 S_1\left(\frac{1}{2}, \frac{1}{2}\right)$	$10^4 S_2\left(\frac{1}{2}, \frac{1}{2}\right)$	$10^4 S_3\left(\frac{1}{2}, \frac{1}{2}\right)$
1	5139 (5139)	7260 (7261)	5997 (5998)
5	1782 (1782)	3688 (3693)	2536 (2536)
10	1005 (1005)	2727 (2737)	1666 (1671)
30	382 (383)	1724 (1750)	859 (878)
50	240 (241)	1399 (1440)	632 (663)

$$- \frac{1}{49(1+49\sqrt{7}d)}, \quad d = c^{-1},$$

$$S_3\left(\frac{1}{2}, \frac{1}{2}\right) = 1{,}1040 - \frac{1}{1+d} - \frac{1}{9\sqrt{3}+243d} - \frac{1}{25\sqrt{5}+3125d}.$$

The accuracy of Eqs. (3.1) can be evaluated on the basis of the calculated results in Table 2. Shown in the table along with exact values of $S_j(1/2, 1/2)$ are approximate values obtained by means of closed solutions (3.1). In the interval $c \in [0; 10]$, the error of Eqs. (3.1) is less than 1%. These equations give exaggerated values of the deflections, moments, and contact forces.

With large c, we approximately sum series (1.8)-(1.10) at the point $x = x_1 = \ell/2$ with the use of the Euler–Maclaurin formula [10]:

$$\sum_{k=1,3\ldots}^{\infty} f(k) \cong \frac{1}{2}\int_1^{\infty} f(x)\,dx - \frac{1}{2}[f(\infty) - f(1)] + \frac{1}{6}[f'(\infty) - f'(1)] - \frac{1}{90}[f'''(\infty) - f'''(1)].$$

We represent its improper integral in the form

$$\int_1^{\infty} f(x)\,dx = \int_0^{\infty} f(x)\,dx - \int_0^{1} f(x)\,dx,$$

where the minuend is the tabulated integral [9]

$$\int_0^{\infty} \frac{x^{\mu-1}\,dx}{p+qx^{\nu}} = \frac{1}{\nu p}\left(\frac{p}{q}\right)^{\mu/\nu} \frac{\pi}{\sin(\pi\mu/\nu)}, \tag{3.2}$$

while the subtrahend is found with any desired accuracy by expanding the integrand into a series in powers of (c^{-1}).

TABLE 3

c	$10^4 S_1\left(\frac{1}{2}, \frac{1}{2}\right)$	$10^4 S_2\left(\frac{1}{2}, \frac{1}{2}\right)$	$10^4 S_3\left(\frac{1}{2}, \frac{1}{2}\right)$
5	1782 (1786)	3688 (3679)	2536 (2531)
10	1005 (1005)	2727 (2726)	1666 (1664)
30	382 (382)	1724 (1725)	859 (859)
50	240 (240)	1399 (1399)	632 (632)
100	126 (126)	1056 (1056)	417 (417)
200	66 (66)	798 (798)	275 (275)
300	45 (45)	678 (678)	216 (216)

TABLE 4

c	$10^4 S_1\left(\frac{1}{2}, \frac{1}{2}\right)$	$10^4 S_2\left(\frac{1}{2}, \frac{1}{2}\right)$	$10^4 S_3\left(\frac{1}{2}, \frac{1}{2}\right)$
5	1828	3470	2515
10	1015	2630	1659
30	383	1695	858
50	240	1382	632
100	126	1047	417
200	66	794	275
300	45	675	216

Let us illustrate the method of summation through the use of an example:

$$S_1\left(\frac{1}{2}, \frac{1}{2}\right) = \sum_{k=1,3\ldots}^{\infty} (k^4 + ck^{3/2})^{-1} = \frac{1}{c} \sum_{k=1,3\ldots}^{\infty} \left(\frac{1}{k^{3/2}} - \frac{k}{k^{5/2}+c}\right) = \frac{1}{c}\left(1{,}689 - \sum_{k=1,3\ldots}^{\infty} \frac{k}{k^{5/2}+c}\right).$$

We have

$$f(x) = x(x^{5/2} + c)^{-1},\ f(1) = (1+c)^{-1},\ f(\infty) = f'(\infty) = f'''(\infty) = 0,$$
$$f'(1) = \frac{1}{2}\frac{2c-3}{(c+1)^2}, \quad f'''(1) = \frac{5}{8}\frac{108c - 21(c^2+1)}{(c+1)^4}.$$

Using Eq. (3.2), we find

$$\int_0^\infty \frac{x\,dx}{x^{5/2}+c} = \frac{2}{5}\frac{1}{c^{1/5}}\frac{\pi}{\sin(4\pi/5)} = \frac{2{,}1379}{c^{1/5}},$$
$$\int_0^1 \frac{x\,dx}{x^{5/2}+c} = \frac{1}{c}\left(\frac{1}{2} - \frac{2}{9c} + \frac{1}{7c^2} - \frac{2}{19c^3}\right) + O(c^{-5}).$$

Having inserted these results into the Euler–MacLaurin formula, we obtain the sum of the series

$$S_1\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{1}{c}\left\{1{,}689 - \frac{1}{2}\left[\frac{2{,}1379}{c^{1/5}} - \frac{1}{c}\left(\frac{1}{2} - \frac{2}{9c} + \frac{1}{7c^2} - 2\frac{1}{19c^3}\right) + \frac{1}{1+c}\right] + \right.$$
$$\left. + \frac{1}{12}\frac{2c-3}{(c+1)^2} + \frac{1}{144}\frac{21(c^2+1) - 108c}{(c+1)^4}\right\}.$$

We similarly find

$$S_2\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{1}{2}\left[\frac{1{,}3213}{c^{2/5}} - \frac{1}{c}\left(\frac{2}{3} - \frac{1}{4c} + \frac{2}{13c^2} - \frac{1}{9c^3}\right) + \frac{1}{1+c}\right] +$$
$$+ \frac{1}{12}\frac{4-c}{(1+c)^2} + \frac{1}{720}\frac{3c^3 - 36c^2 + 519c - 192}{(c+1)^4}, \tag{3.3}$$

$$S_3\left(\frac{1}{2},\frac{1}{2}\right)=\frac{1}{2}\left[\frac{1{,}3213}{c^{3/5}}-\frac{1}{c}\left(1-\frac{2}{7c}+\frac{1}{6c^2}-\frac{2}{17}\frac{1}{c^3}\right)+\frac{1}{1+c}\right]+\frac{5}{12}\frac{1}{(1+c)^2}+$$
$$+\frac{1}{144}\frac{84c-3c^2-63}{(c+1)^4}.$$

The numbers in parentheses in Table 3 show the results of calculations performed with Eqs. (3.3) and exact values of $S_j(1/2, 1/2)$. It can be seen that at, $c > 5$, closed solutions (3.3) ensure good accuracy in the calculation (error smaller than 1%). Closed solutions (3.1), (3.3) include the entire range of possible values of c and make numerical summation of the series unnecessary.

At large c, Eqs. (3.3) can be simplified by discarding the second-degree terms. This operation gives

$$S_1\left(\frac{1}{2},\frac{1}{2}\right)=c^{-1}(1{,}689-1{,}069c^{-1/5}),$$
$$S_2\left(\frac{1}{2},\frac{1}{2}\right)=0{,}6607c^{-2/5},\quad S_3\left(\frac{1}{2},\frac{1}{2}\right)=0{,}6607c^{-3/5}. \tag{3.4}$$

Calculations performed with Eqs. (3.4) show (see Table 4) that their error is less than 2% within the interval $30 \le c \le 300$. Thus, for large c, the calculation of the maximum deflections, bending moments, and contact stresses reduces to the elementary formulas

$$w\left(\frac{l}{2},\frac{l}{2}\right)=\frac{2Pl^3}{\pi^4E_1I_1c}(1{,}689-1{,}069c^{-1/5}),$$
$$M\left(\frac{l}{2},\frac{l}{2}\right)=1{,}3213\frac{Pl}{\pi^2c^{2/5}},\quad q\left(\frac{l}{2},\frac{l}{2}\right)=1{,}3213\frac{Pc^{2/5}}{l}.$$

In conclusion, we emphasize that the analytical results obtained here pertain to the case of loading of a reinforced shell by a single radial force. When the ribs are acted upon by a cyclically symmetric load comprising several concentrated stresses, the simplifications that were made lose meaning and it becomes necessary to construct periodic solutions in the angular coordinate — as has been done in [11, 12] and other studies in the theory of ribbed shells.

LITERATURE CITED

1. I. F. Obraztsov, B. V. Nerubaiko, and V. P. Ol'shanskii, "Shells subjected to local loads (survey of studies, main results, and research trends)," Submitted to VINITI 12.02.88, No. 1222, Moscow (1988).
2. A. L. Gol'denveizer, "Design of shells to withstand concentrated forces," Prikl. Mat. Mekh., 18, No. 2 (1954).
3. G. N. Chernyshev, "Contact problems in shell theory," Transactions of the 7th All-Union Conference on the Theory of Plates and Shells [in Russian], Nauka, Moscow (1970).
4. F. K. G. Odquist, "Action of forces and moments symmetrically distributed along a generatrix of thin cylindrical shells," J. Appl. Mech., 13, No. 2 (1946).
5. H. Schores, "Line loads action on thin cylindrical shells," Proc. Am. Soc. Civ. Eng., 61, No. 3 (1935).
6. G. Bateman and A. Erdelyi, Tables of Integral Transforms. Fourier, Laplace, and Mellin Transforms [Russian translation], Nauka, Moscow (1969).
7. E. Yanke, F. Emde, and F. Lesh, Special Functions [in Russian], Nauka, Moscow (1977).
8. G. Bateman and A. Erdelyi, Higher Transcendental Functions [in Russian], Vol. 1, Nauka, Moscow (1973).
9. I. S. Gradshtein and I. M. Ryzhik, Tables of Integrals, Sums, Series, and Products [in Russian], Fizmatgiz, Moscow (1962).
10. G. M. Fikhtengol'ts, Course in Differential and Integral Calculus [in Russian], Vol. 2, Nauka, Moscow (1969).
11. É. I. Grigolyuk and V. M. Tolkachev, Contact Problems of the Theory of Plates and Shells [in Russian], Mashinostroenie, Moscow (1980).
12. V. N. Maksimenko and L. A. Fil'shtinskii, "Elastic equilibrium of anisotropic shells reinforced by ribs," Prikl. Mat. Mekh., 39, No. 5 (1976).